MATH 1A – QUIZ 4 – SOLUTIONS

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(1) *(3 points)* Using the **definition** of the derivative, calculate the derivative of $f(x) = \cos(x)$. You may use any limits we talked about in section.

Hint: $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} && \text{(By the hint)} \\
&== \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x) - \sin(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \frac{\sin(x)\sin(h)}{h} \\
&= \cos(x)\left(\lim_{h\to 0} \frac{\cos(h) - 1}{h}\right) - \sin(x)\left(\lim_{h\to 0} \frac{\sin(h)}{h}\right) \\
&= \cos(x)(0) - \sin(x)(1) && \text{(By the 'special limits' we talked about in class)} \\
&= -\sin(x)
\end{aligned}
$$

Hence $\boxed{f'(x) = -\sin(x)}$

(2) *(1 point)* Is the function $f(x) = \sqrt{x}$ differentiable at 0 (from the right)? Explain.

$$
\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{\sqrt{x} - 0}{x} = \lim_{x\to 0^+} \frac{\sqrt{x}}{x} = \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty
$$

Since $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$ does not exist, f is not differentiable at 0.

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(3) *(2 points; 1 point each)* Evaluate the following limits:

Note: Again, $-\infty$ points for using l'Hopital's rule!

(a)

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^6+x^2+1}}{x^3} &= \lim_{x\to-\infty} \frac{\sqrt{x^6\left(1+\frac{1}{x^4}+\frac{1}{x^6}\right)}}{x^3} \\
&= \lim_{x\to-\infty} \frac{\sqrt{x^6}\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}}}{x^3} \\
&= \lim_{x\to-\infty} \frac{\left|x^3\right|\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}}}{x^3} \\
&= \lim_{x\to-\infty} \frac{-\cancel{x^3}\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}}}{\cancel{x^3}} \\
&= \lim_{x\to-\infty} -\sqrt{1+\frac{1}{x^4}+\frac{1}{x^6}} \\
&= -\sqrt{1+0+0} \\
&= -1
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln(x))^2-1}{(\ln(x))^2-3} &= \lim_{x\to\infty} \frac{\cancel{(\ln(x))^2}\left(1-\frac{1}{(\ln(x))^2}\right)}{\cancel{(\ln(x))^2}\left(1-\frac{3}{(\ln(x))^2}\right)} \\
&= \lim_{x\to\infty} \frac{\left(1-\frac{1}{(\ln(x))^2}\right)}{\left(1-\frac{3}{(\ln(x))^2}\right)} = \qquad \frac{1-\frac{1}{\infty}}{1-\frac{3}{\infty}} \\
&= \frac{1-0}{1-0} \\
&= 1
\end{aligned}$$

Note: From now on, you're allowed to use differentiation formulas!

(4) *(1 point)* Find $f'(x)$, where $f(x) = \frac{e^x}{\cos(x)}$

$$f'(x) = \frac{e^x\cos(x) - e^x\left(-\sin(x)\right)}{\left(\cos(x)\right)^2} = \frac{e^x\left(\cos(x) + \sin(x)\right)}{\left(\cos(x)\right)^2}$$

(5) *(3 points)* Find the equation of the tangent line to $f(x) = \sqrt{x}$ whose $x-$ intercept is -4

The equation of the tangent line to f at a point a is $y - f(a) = f'(a)(x - a)$, which in this case becomes $y - \sqrt{a} = \frac{1}{2\sqrt{a}}(x - a)$

Now we want that tangent line to have $x-$ intercept -4, that is we want $(-4, 0)$ to be on that tangent line, hence we get:

$$\begin{aligned} 0 - \sqrt{a} =& \frac{1}{2\sqrt{a}}(-4 - a) \\ (-\sqrt{a})(2\sqrt{a}) =& -4 - a \\ -2a =& -4 - a \\ -a =& -4 \\ a =& 4 \end{aligned}$$

Hence, the equation of the tangent line becomes: $y - \sqrt{4} = \frac{1}{2\sqrt{4}}(x - 4)$, that is: $\boxed{y - 2 = \frac{1}{4}(x - 4)}$, or $\boxed{y = \frac{x}{4} + 1}$ (either of those last 2 answers is fine)